Heat Transfer: Conduction

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INTRODUCTION TO HEAT TRANSFER

What is Heat Transfer?

Energy in transit due to a temperature difference.

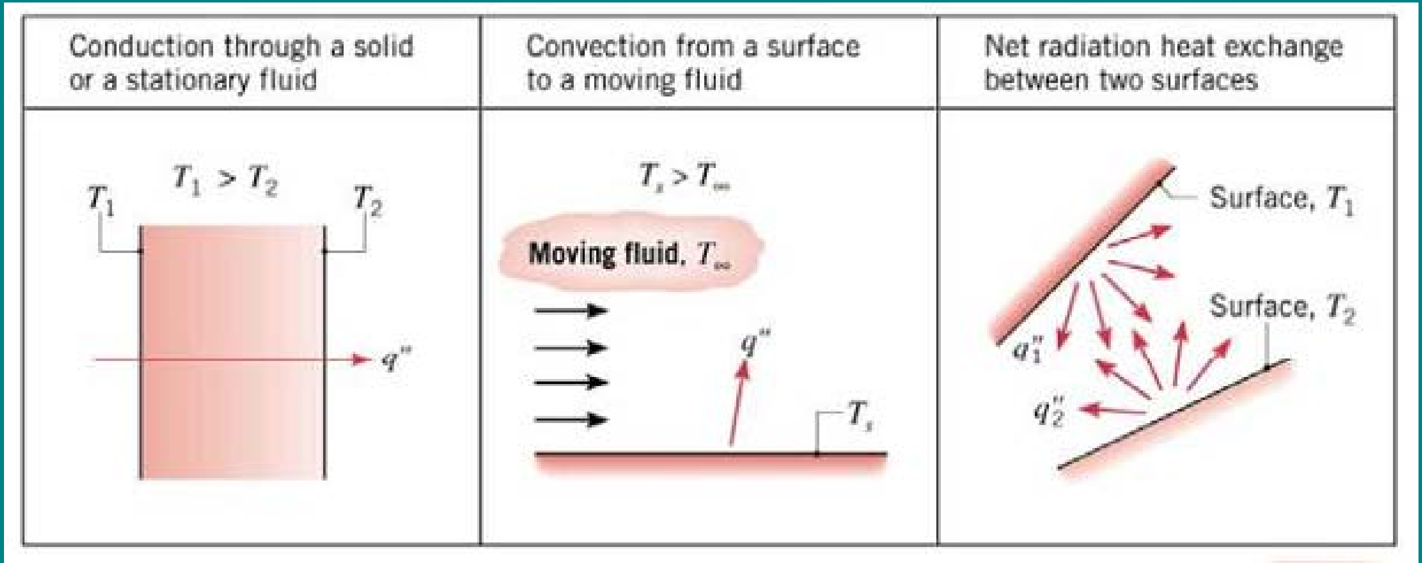
How is heat transferred?

Conduction through a solid or a stationary fluid

Convection from a surface to a moving fluid (or vice versa)

Net *radiation* heat exchange between two or more surfaces.

INTRODUCTION TO HEAT TRANSFER



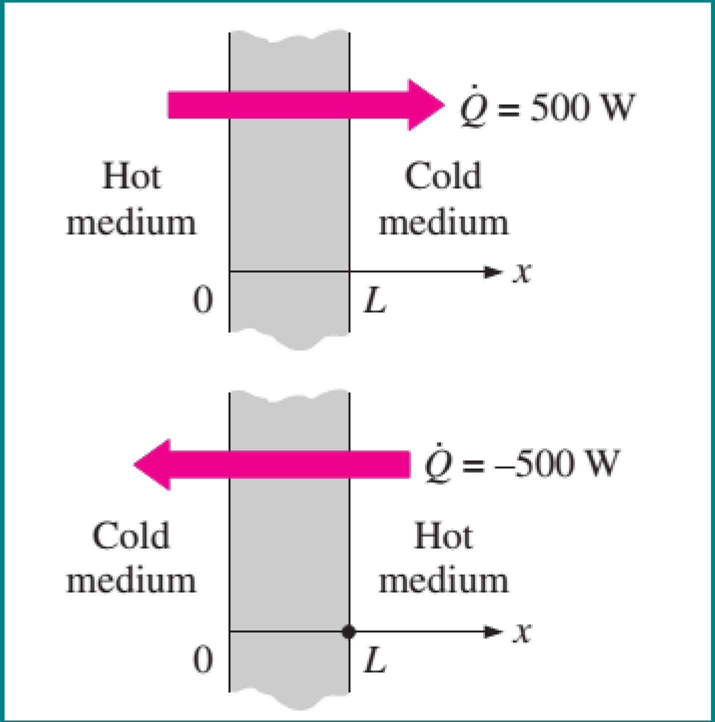
Source: Incropera *et al.* (2007)

Our focus *Our focus*

CONDUCTION

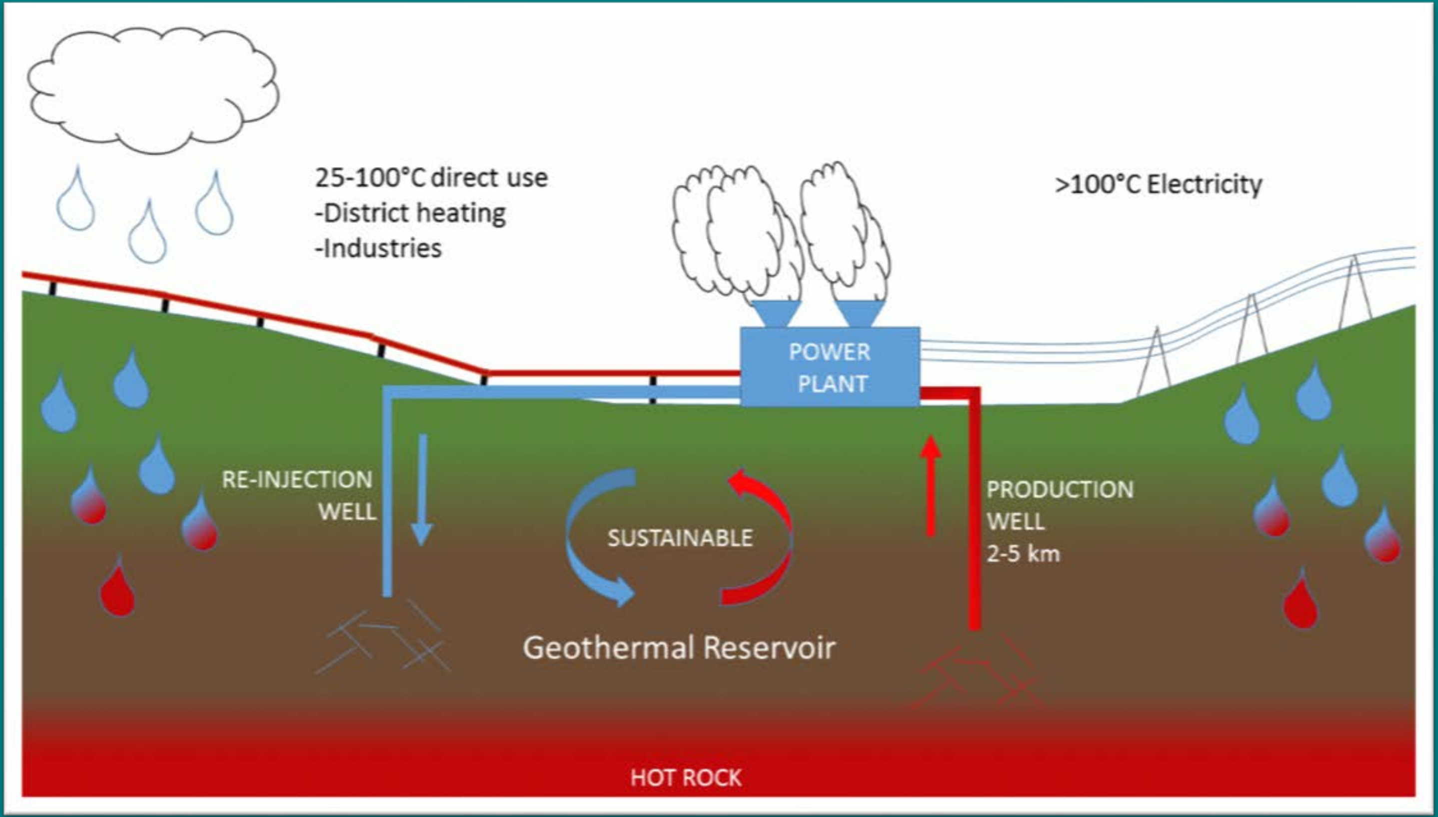
Physical Origins and Rate Equations

- Transfer of energy from more energetic particles of a substance to adjacent less energetic ones as a result if interactions between the particles
- No net motion of material (only diffusion of energy)
- Can occur in solids, stationary liquids and gases
- Three-dimensional in nature



Source: Cengel et al. 2010

GEOTHERMAL CYCLES



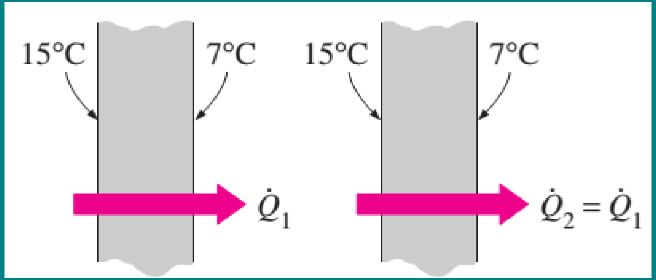
Source: arcticgreencorp.com

MECHANISMS OF HEAT CONDUCTION

- In a solid, the flow of heat by conduction is the result of the transfer of vibrational energy from one molecule to another.
- In fluids: it occurs in additional as a result of the transfer of kinetic energy.
- Heat transfer by conduction may also arise from the movement of free electrons, a process which is particularly important with metals and accounts for their high thermal conductivities.
- It is need a medium to transfer, and moves from high region to low region

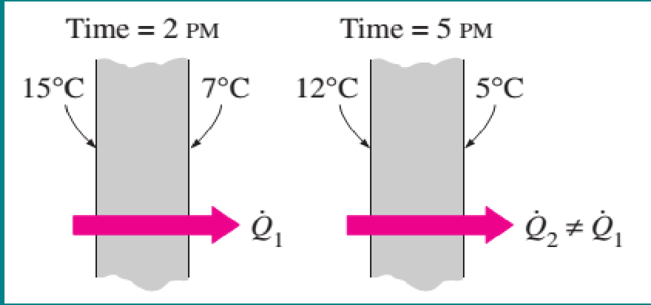
STEADY AND TRANSIENT HEAT TRANSFER

- Steady (steady-state): no change with time at any point within the medium



Source: Cengel et al. 2010

- Transient(unsteady): implies variation with timeor time dependence.



Source: Cengel et al. 2010

MULTIDIMENSIONAL HEAT TRANSFER

Heat transfer problems are classified as being *one-dimensional, two-dimensional,* or *three-dimensional* (depending on the relative magnitudes and the level of accuracy). In the most general case, heat transfer through a medium is **three-dimensional**.

The temperature distribution throughout the medium:

Cartesian Coordinates $T(x, y, z)$

Cylindrical Coordinates $T(r, \emptyset, z)$

Spherical Coordinates $T(r, \varphi, \theta)$

HEAT TRANSFER RATE OF CONDUCTION

Fourier's Law

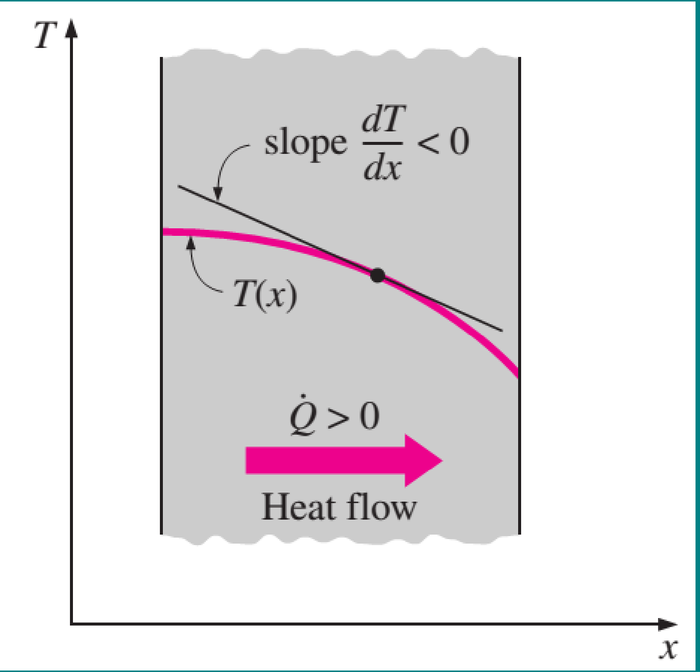
The rate of heat conduction through a medium (for example in the x-direction) is proportional to the temperature difference across the medium and the area normal to the direction.

$$\dot{Q}_{cond} = -kA\frac{dT}{dx}$$

where

k is thermal conductivityof the material

$\frac{dT}{dx}$ is the temperature gradient



Source: Cengel et al. 2010

The heat flux vector at a point P on this surface must be perpendicular to the surface, and it must point in the direction of decreasing temperature.

The rate of heat conduction at that point can be expressed by Fourier's law as:

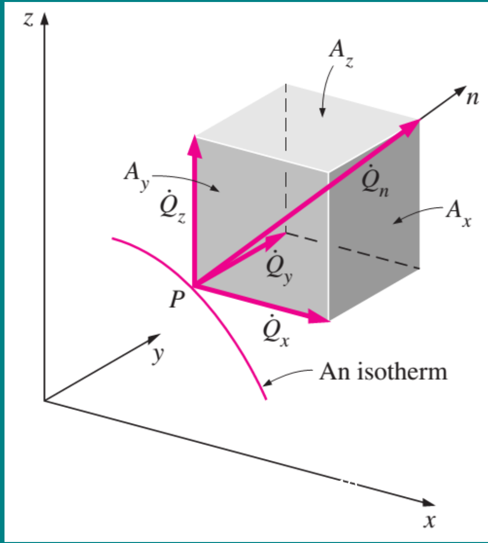
$$\dot{Q}_n = -kA\frac{\partial T}{\partial x} \qquad (W)$$

In rectangular coordinates:

$$\overrightarrow{\dot{Q}_n} = \dot{Q}_x\vec{\boldsymbol{i}} + \dot{Q}_y\vec{\boldsymbol{j}} + \dot{Q}_z\vec{\boldsymbol{k}}$$

It can be determined from Fourier's law as:

$$\dot{Q}_x = -kA_x\frac{\partial T}{\partial x},\ \dot{Q}_y = -kA_y\frac{\partial T}{\partial y},\ \text{and}\ \dot{Q}_x = -kA_z\frac{\partial T}{\partial z}$$



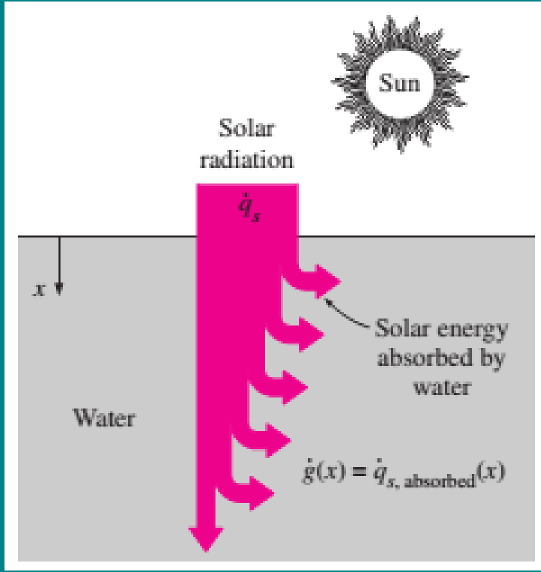
Source: Cengel et al. 2010

HEAT GENERATION

Heat Generation: A medium through which heat is conducted may involve the conversion of electrical, nuclear, or chemical energy into heat (or thermal) energy

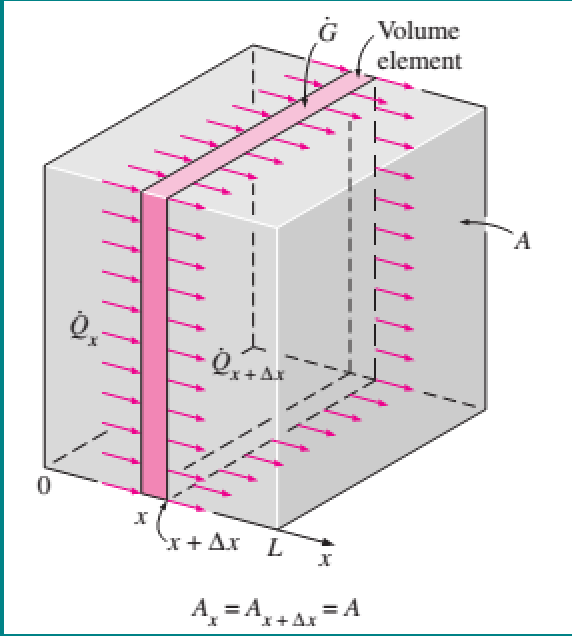
The rate of heat generation in a medium may vary with time as well as position within the medium. The total rate of heat generation in a medium of volume V can be determined from:

$$\dot{G} = \int_v g\, dV \quad (W)$$



Source: Cengel et al. 2010

1-D HEAT CONDUCTION EQUATION



Source: Cengel et al. 2010

Heat conduction in can be approximated as being one-dimensional since heat conduction through these geometries will be dominant in one direction and negligible in other directions.

<u>Heat Conduction Equation in a Large Plane Wall</u>

An energy balance on this thin element during a small time interval ΔT be expressed as:

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{G}_{element} = \frac{\Delta E_{element}}{\Delta T}$$

1-D HEAT CONDUCTION EQUATION

The one-dimensional transient heat conduction equation in a plane wall:

Variable conductivity: $$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\dot{g}=\rho C\frac{\partial T}{\partial t}$$

The thermal conductivity k of a material depends on the temperature, and thus it cannot be taken out of the derivative. However, the thermal conductivity in most practical applications can be assumed to remain constantat some average value. The equation above in that case reduces to:

Constant conductivity: $$\frac{\partial^2 T}{\partial x^2}+\frac{\dot{g}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Where α is the thermal diffusivity, $\alpha=\frac{k}{\rho C}$

1-D HEAT CONDUCTION EQUATION

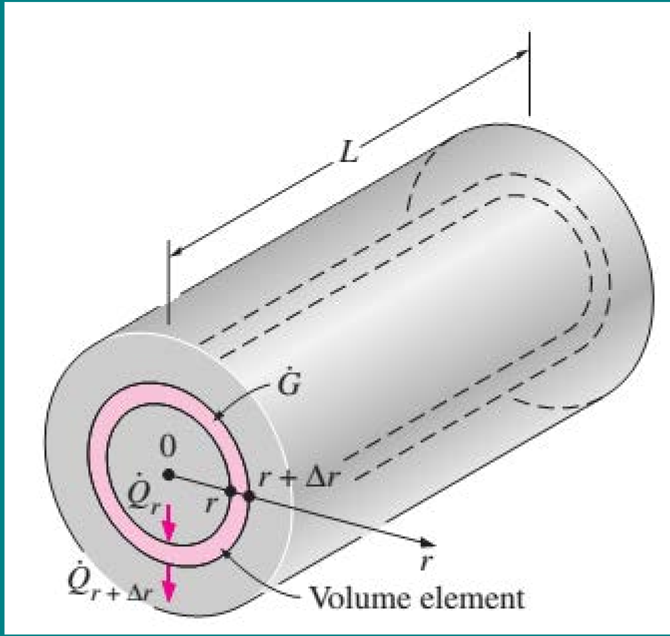
For the following forms under specified conditions for a Large Plane Wall :

1. Steady-state $\left(\frac{\partial}{\partial t} = 0\right)$: $$\frac{d^2T}{dx^2} + \frac{\dot{g}}{k} = 0$$
2. Transient, no heat generation $(\dot{g} = 0)$: $$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$
3. Steady-state, no heat generation $\left(\frac{\partial}{\partial t} = 0 \; and \; \dot{g} = 0\right)$: $$\frac{d^2T}{dx^2} = 0$$

1-D HEAT CONDUCTION EQUATION

Heat Conduction Equation in a Long Cylinder

Consider a thin cylindrical shell element of thickness Δr in a long cylinder.



Source: Cengel et al. 2010

Assume the density of the cylinder is ρ, the specific heat is C, and the length is L.

The one-dimensional transient heat conduction equation in a a cylinder:

Variable conductivity:
$$\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) + \dot{g} = \rho C\frac{\partial T}{\partial t}$$

The equation above in that case reduces to:

Constant conductivity:
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

1-D HEAT CONDUCTION EQUATION

For the following forms under specified conditions for a Cylinder:

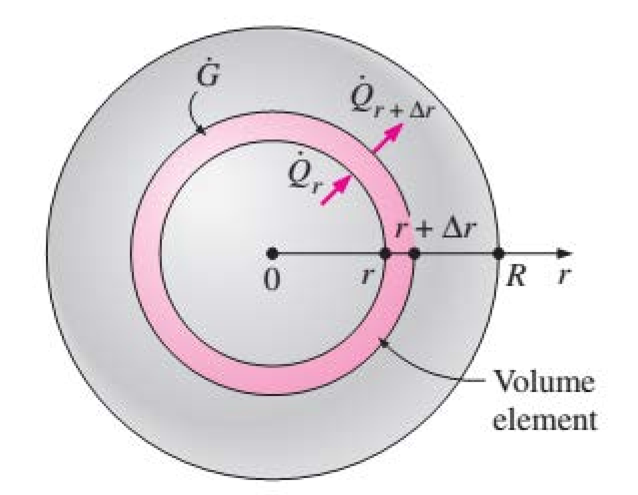
1. Steady-state $\left(\frac{\partial}{\partial t} = 0\right)$: $$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\dot{g}}{k} = 0$$
2. Transient, no heat generation $(\dot{g} = 0)$: $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$
3. Steady-state, no heat generation : $$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

   $\left(\frac{\partial}{\partial t} = 0 \; and \; \dot{g} = 0\right)$

1-D HEAT CONDUCTION EQUATION

Heat Conduction Equation in a Sphere

Consider a sphere with density ρ, specific heat C, and outer radius R.



Source: Cengel et al. 2010

The one-dimensional transient heat conduction equation in a sphere:

Variable conductivity: $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k \frac{\partial T}{\partial r}\right) + \dot{g} = \rho C \frac{\partial T}{\partial t}$$

The equation above in that case reduces to:

Constant conductivity: $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

1-D HEAT CONDUCTION EQUATION

For the following forms under specified conditions for a Sphere :

1. Steady-state $\left(\frac{\partial}{\partial t} = 0\right)$: $$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) + \frac{\dot{g}}{k} = 0$$
2. Transient, no heat generation $(\dot{g} = 0)$: $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$
3. Steady-state, no heat generation : $\left(\frac{\partial}{\partial t} = 0 \; and \; \dot{g} = 0\right)$ $$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0, \text{ or}$$ $$\mathrm{r}\frac{d^2T}{dr^2} + 2\frac{dT}{dr} = 0$$

GENERAL HEAT CONDUCTION EQUATION

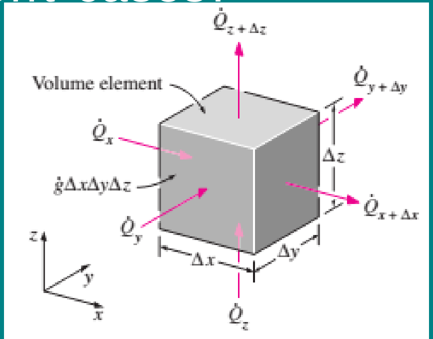
From **Fourier-Biot** equation $\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}\right)$, we can determine general heat conduction equation for different cases:

1. Rectangular Coordinates

   a. Steady-state (**Poisson equation**) : $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{g}}{k} = 0$

   b. Transient, no heat generation (**diffusion equation**) : $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$

   c. Steady-state, no heat generation (**Laplace equation**) : $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$



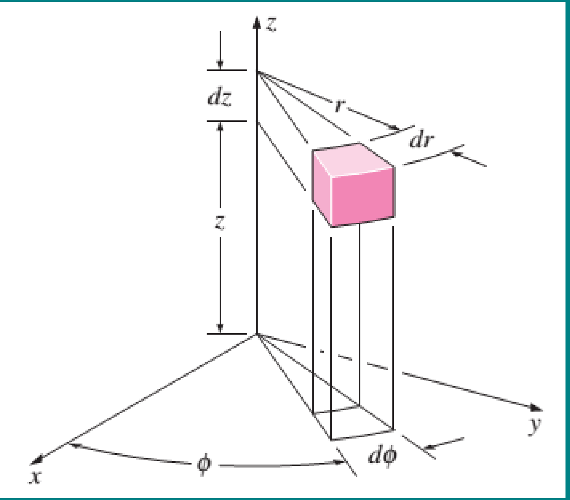
Source: Cengel et al. 2010

GENERAL HEAT CONDUCTION EQUATION

2. Cylindrical Coordinates

Using the following relations between the coordinates of a point in rectangular and cylindrical coordinate systems:

$$x = r\,cos\emptyset, \quad y = \mathrm{r}\,\sin\emptyset, \quad \text{and } z = z$$



Source: Cengel et al. 2010

The general heat conduction equation in cylindrical coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \emptyset}\left(kr\frac{\partial T}{\partial \emptyset}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{g} = \rho C\frac{\partial T}{\partial t}$$

GENERAL HEAT CONDUCTION EQUATION

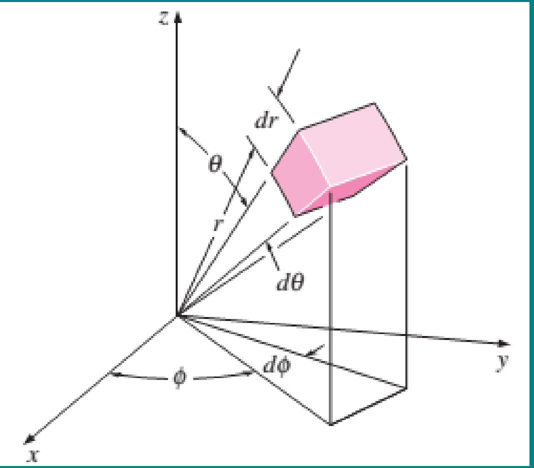
3. Spherical Coordinates

Using the following relations between the coordinates of a point in rectangular and spherical coordinate systems:

$$x = r\, cos\emptyset\, sin\theta, \quad \mathrm{y} = \mathrm{r}\,\sin\emptyset\, sin\theta, \quad \text{and } z = r\, cos\theta$$

The general heat conduction equation in spherical coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2 sin^2\emptyset}\frac{\partial}{\partial \emptyset}\left(k\frac{\partial T}{\partial \emptyset}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta}\left(k\, sin\theta\frac{\partial T}{\partial \theta}\right) + \dot{g} = \rho C\frac{\partial T}{\partial t}$$



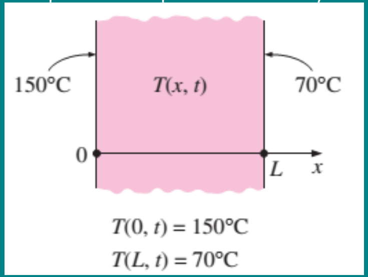
Source: Cengel et al. 2010

BOUNDARY AND INITIAL CONDITIONS OF CONVECTION

1. Specified Temperature Boundary Condition

$$T(0,t) = T_1$$
$$T(L,t) = T_2$$

Specified Temperature Boundary

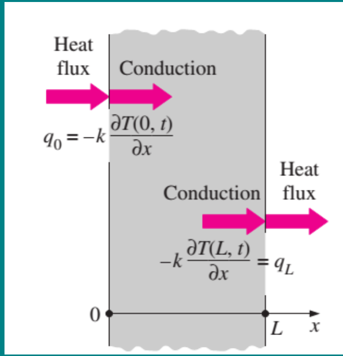


Source: Cengel et al. 2010

2. Specified Heat Flux Boundary Condition

$$q = -k\frac{\partial T}{\partial x} = \left(\begin{matrix} Heat\ flux\ in\ the \\ positive\ x - direction \end{matrix}\right)(W/m^2)$$

Specified Heat Flux Boundary

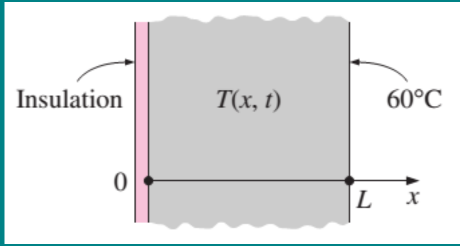


Source: Cengel et al. 2010

Special Case: Insulated Boundary

$$C = 0 \text{ or } \frac{\partial T(0.t)}{\partial x} = 0$$

Insulated Boundary

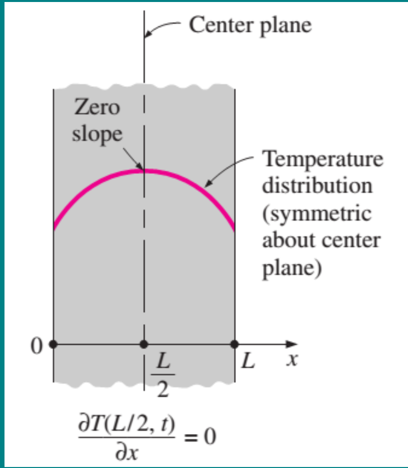


Source: Cengel et al. 2010

Another Special Case: Thermal Symmetry

$$\frac{\partial T(\frac{L}{2}, t)}{\partial x} = 0$$

Thermal Symmetry



Source: Cengel et al. 2010

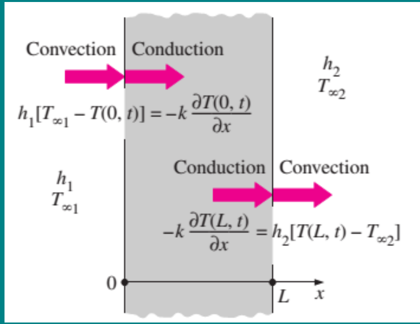
3. Convection Boundary Condition

$$-k\frac{\partial T(0,t)}{\partial x} = h_1[T_{\infty 1} - T(0,t)]$$

and

$$-k\frac{\partial T(L,t)}{\partial x} = h_2[T(L,t) - T_{\infty 2}]$$

Convection boundary conditions



Source: Cengel et al. 2010

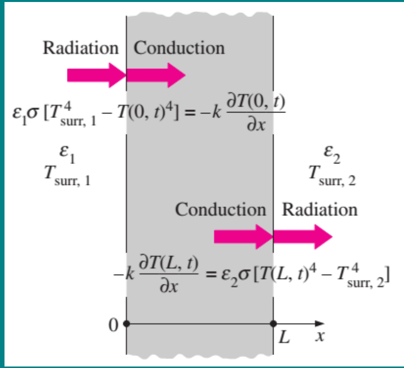
4. Radiation Boundary Condition

$$-k\frac{\partial T(0,t)}{\partial x} = \varepsilon_1\sigma[T_{surr,1}^4 - T(0,t)^4]$$

and

$$-k\frac{\partial T(L,t)}{\partial x} = \varepsilon_2\sigma[T(L,t)^4 - T_{surr,2}^4]$$

Radiation boundary conditions



Source: Cengel et al. 2010

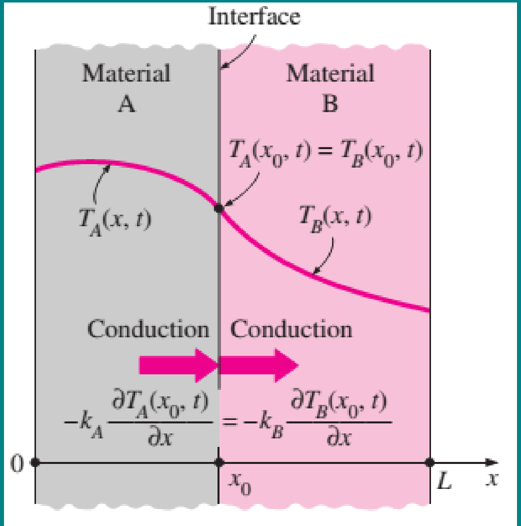
5. Interface Boundary Conditions

$$T_A(x_0, t) = T_{\boldsymbol{A}}(x_0, t)$$

And

$$-k_{\boldsymbol{A}} \frac{\partial T_A(x_0,t)}{\partial x} = -k_{\boldsymbol{B}} \frac{\partial T_B(x_0,t)}{\partial x}$$

The interface of two bodies in perfect contact



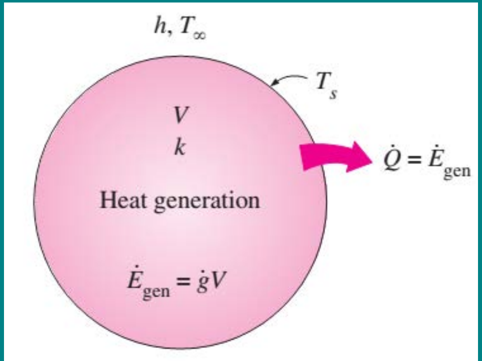
Source: Cengel et al. 2010

6. Generalized Boundary Conditions

$$\begin{pmatrix} Heat\ transfer \\ to\ the\ surface \\ in\ all\ modes \end{pmatrix} = \begin{pmatrix} Heat\ transfer \\ from\ the\ surface \\ in\ all\ modes \end{pmatrix}$$

HEAT GENERATION IN A SOLID

A solid medium of surface area A_s , volume V, and constant thermal conductivity k , where heat is generated at a constant rate of $\dot{g}$ per unit volume. Heat is transferred from the solid to the surrounding medium at T , with a constant heat transfer coefficient of h . All the surfaces of the solid are maintained at a common temperature T_s.



Source: Cengel et al. 2010

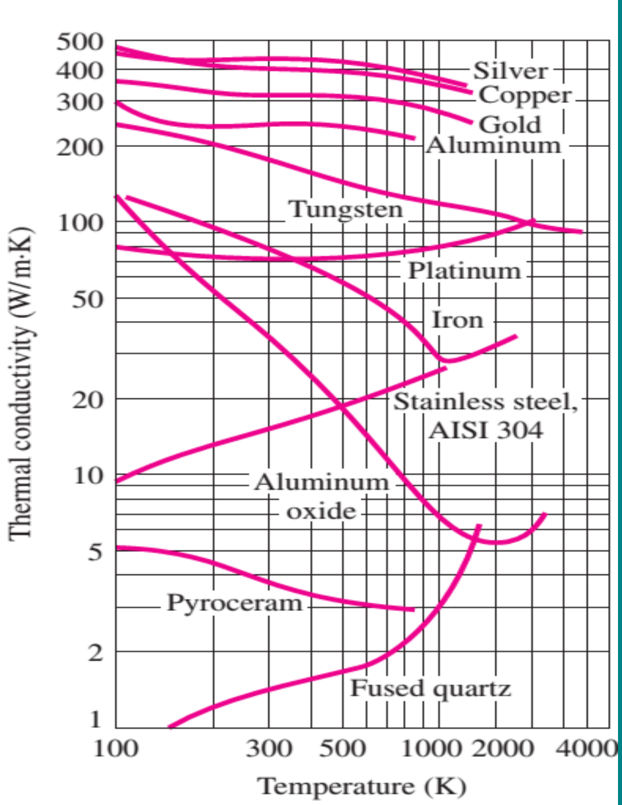
Under steady conditions, the energy balance for this solid can be expressed as:

$$\dot{Q} = \dot{g}V = hA_s(T_s - T_\infty)$$

The surface temperature T_s gives:

$$T_s = T_\infty + \frac{\dot{g}V}{hA_s}$$

VARIABLE THERMAL CONDUCTIVITY, $k(T)$



Source: Cengel et al. 2010

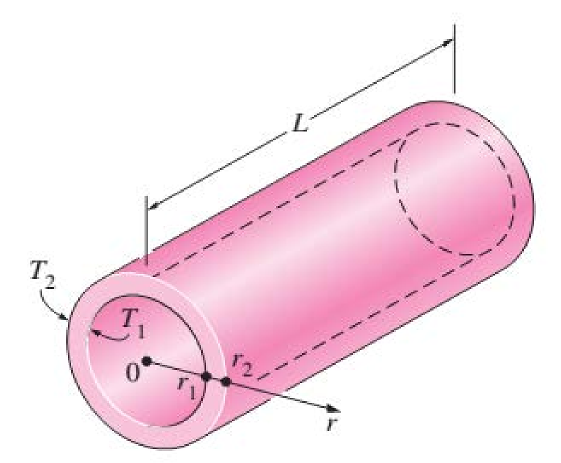
When the variation of thermal conductivity with temperature $k(T)$ is known, the average value of the thermal conductivity in the temperature range between T_1 and T_2 can be determined from:

$$k_{ave} = \frac{\int_{T_1}^{T_2} k(T)dT}{T_2 - T_1}$$

EXAMPLE

Heat Loss through a Steam Pipe

Consider a steam pipe of length $L = 20\,m$, inner radius $r_1 = 6\,cm$, outer radius $r_2 = 8\,cm$, and thermal conductivity $k = 20\,W/m.℃$. The inner and outer surfaces of the pipe are maintained at average temperatures of $T_1 = 150℃$ and $T_1 = 60℃$, respectively. Obtain a general relation for the temperature distribution inside the pipe under steady conditions, and determine the rate of heat loss from the steam through the pipe.



Source: Cengel et al. 2010

Assumptions : 1. Steady, 2. One-dimensional, 3. Thermal conductivity is constant, 4. No heat generation.

Properties : The thermal conductivity is given to be $k = 20\,W/m.℃$.

Analysis :

The mathematical formulation of this problem: $\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$

Boundary conditions: $T(r_1) = T_1 = 150℃$ and $T(r_2) = T_2 = 60℃$

Integrating the differential equation once with respect to r gives:

$$r\frac{dT}{dr} = C_1 \longrightarrow \frac{dT}{dr} = \frac{C_1}{r}$$

Again integrating with respect to r:

$$T(r) = C_1 \ln r + C_2$$

We now apply both boundary conditions

$$T(r_1) = T_1 \longrightarrow C_1 \ln r_1 + C_2 = T_1$$

$$T(r_2) = T_2 \longrightarrow C_1 \ln r_2 + C_2 = T_2$$

Analysis :

There are two equations in two unknowns, $C_1\ and\ C_2$. Solving them simultaneously gives

$$C_1 = \frac{T_2 - T_1}{\ln\left(\frac{r_2}{r_1}\right)} \text{ and } C_2 = T_1 - \frac{T_2 - T_1}{\ln\left(\frac{r_2}{r_1}\right)} \ln r_1$$

The variation of temperature within the pipe is determined to be

$$T(r) = C_1 \ln r + C_2$$

$$T(r) = \left(\frac{\ln\left(\frac{r}{r_1}\right)}{\ln\left(\frac{r_2}{r_1}\right)}\right)(T_2 - T_1) + T_1$$

Analysis :

The rate of heat loss from the steam is simply the total rate of heat conduction through the pipe, and is determined from Fourier's law to be

$$\dot{Q}_{cylinder} = -\mathrm{kA}\frac{dT}{dr}$$

$$= -k(2\pi rL)\,\frac{C_1}{r}$$

$$= 2\pi kL\frac{T_1 - T_2}{\ln\left(\frac{r_2}{r_1}\right)} = 2\pi(20\,W/m\,.℃)(20\,m)\frac{(150-60)℃}{\ln\left(\frac{0.08}{0.06}\right)}$$

$$\dot{Q} = 786\,kW$$

Discussion : The total rate of heat transfer through a pipe is constant, but the heat flux is not since it decreases in the direction of heat transfer with increasing radius since $\dot{q} = \dot{Q}/(2\pi rL)$.

Steady Heat Conduction

1. Steady Heat Conduction in Plane Walls

The energy balance for the wall can be expressed as:

$$\begin{pmatrix} \text{Rate of heat transfer} \\ \text{into the wall} \end{pmatrix} - \begin{pmatrix} \text{Rate of heat transfer} \\ \text{out of the wall} \end{pmatrix} = \begin{pmatrix} \text{Rate of change of} \\ \text{the energy of the wall} \end{pmatrix}$$

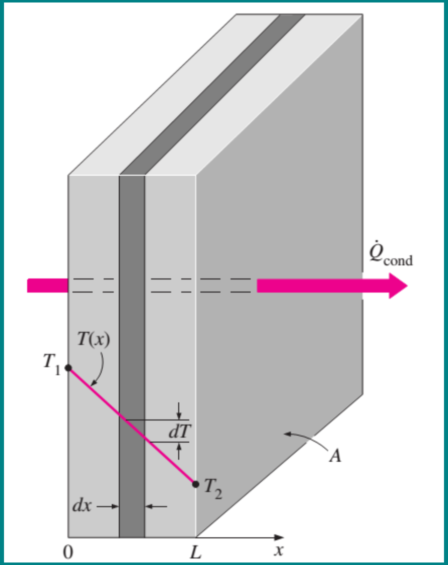
$$\dot{Q}_{in} - \dot{Q}_{out} = \frac{dE_{wall}}{\mathrm{dt}}$$

for steady operation $\frac{dE_{wall}}{\mathrm{dt}} = 0$. Then the rate of heat transfer through the wall must be constant, $\dot{Q}_{cond,wall} = constant$.

$$\dot{Q}_{cond,wall} = -\mathrm{kA}\frac{dT}{dx} = \mathrm{kA}\frac{T_1 - T_2}{L} \quad (\mathrm{W})$$

If R_{wall} is <u>thermal resistance</u> of the wall against heat conduction or simply the conduction resistance of the wall

$$R_{wall} = \frac{L}{\mathrm{kA}} \; (°\mathrm{C/W})$$

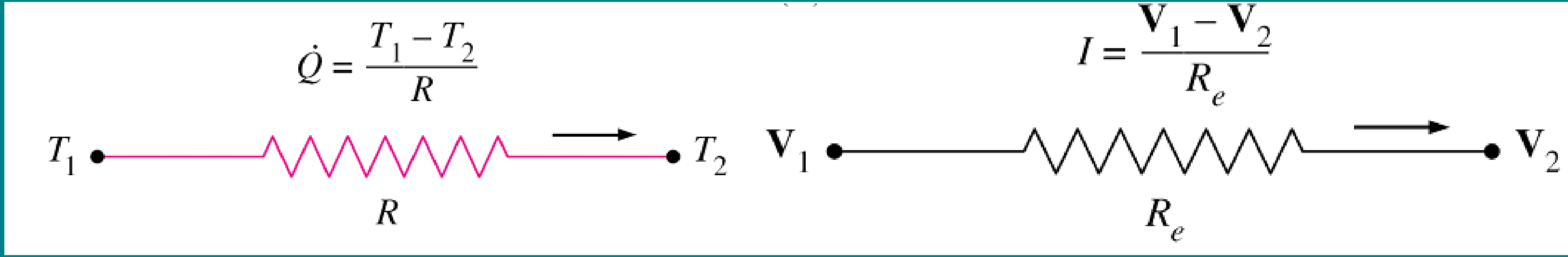


Source: Cengel et al. 2010

The rate of heat transfer through the wall can be described as:

$$\dot{Q}_{cond,wall} = \frac{T_1 - T_2}{R_{wall}} \text{ (W)}$$

- Analogy to Electrical Current Flow:



<u>Heat Transfer</u>		<u>Electrical current flow</u>
Rate of heat transfer	⟷	Electric current
Thermal resistance	⟷	Electrical resistance
Temperature difference	⟷	Voltage difference

THERMAL RESISTANCE

Total thermal resistance:

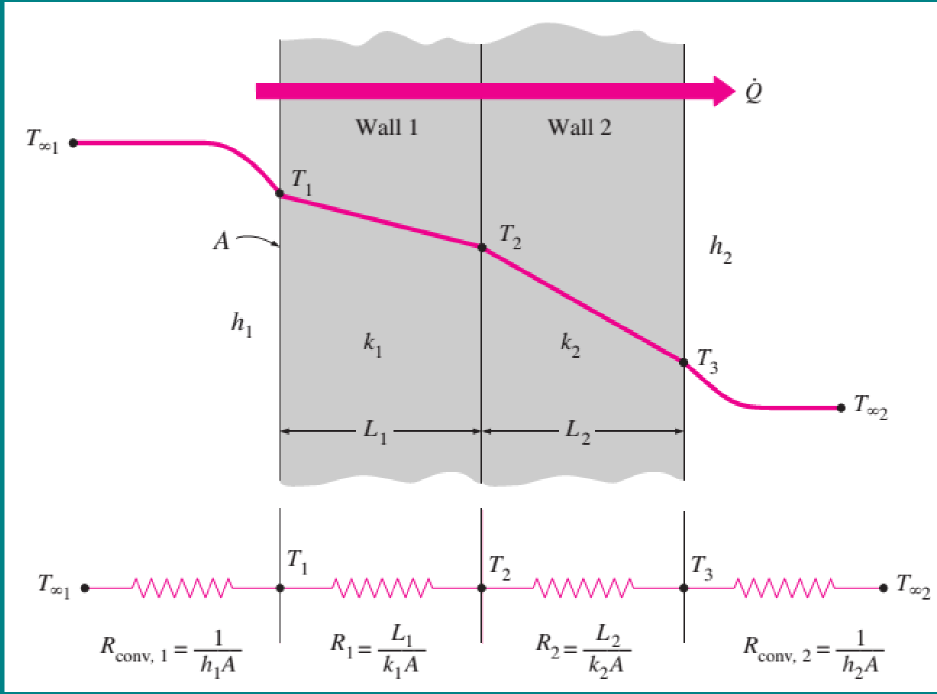
$$R_{total} = R_{conv,1} + R_1 + R_2 + R_{conv,2}$$

For a unit area, overall heat transfer coefficient U is the inverse of the total thermal resistance:

$$UA = \frac{1}{R_{total}}$$

Newton's law of cooling:

$$\dot{Q} = UA\,\Delta T$$



Source: Cengel et al. 2010

THERMAL CONTACT CONDUCTANCE

Thermal contact conductance of some metal surfaces in air (from various sources)

Material	Surface Condition	Roughness, μm	Temperature, °C	Pressure, MPa	h_c,* W/m² · °C
Identical Metal Pairs					
416 Stainless steel	Ground	2.54	90–200	0.3–2.5	3800
304 Stainless steel	Ground	1.14	20	4–7	1900
Aluminum	Ground	2.54	150	1.2–2.5	11,400
Copper	Ground	1.27	20	1.2–20	143,000
Copper	Milled	3.81	20	1–5	55,500
Copper (vacuum)	Milled	0.25	30	0.7–7	11,400
Dissimilar Metal Pairs					
Stainless steel–				10	2900
Aluminum		20–30	20	20	3600
Stainless steel–				10	16,400
Aluminum		1.0–2.0	20	20	20,800
Steel Ct-30–				10	50,000
Aluminum	Ground	1.4–2.0	20	15–35	59,000
Steel Ct-30–				10	4800
Aluminum	Milled	4.5–7.2	20	30	8300
				5	42,000
Aluminum-Copper	Ground	1.3–1.4	20	15	56,000
				10	12,000
Aluminum-Copper	Milled	4.4–4.5	20	20–35	22,000

*Divide the given values by 5.678 to convert to Btu/h · ft² · °F.

Source: Cengel et al. 2010

HEAT CONDUCTION IN CYLINDERS

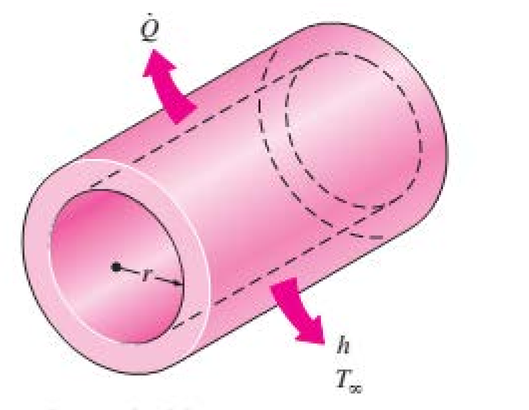
Fourier's law of heat conduction for heat transfer through the cylindrical layer can be expressed as:

$$\dot{Q}_{cond,cyl} = -kA\frac{dT}{dr}$$

$$\dot{Q}_{cond,cyl} = \frac{T_1 - T_2}{R_{cyl}} \qquad (W)$$

Thermal resistance of the cylindrical layer:

$$R_{cyl} = \frac{\ln(r_2/r_1)}{2\pi Lk} = \frac{\ln(\text{Outer radius}/\text{Inner radius})}{2\pi\ (Length)\ (Thermal\ conductivity)}$$



Heat is lost from a hot water pipe to the air outside in the radial direction, and thus heat transfer from a long pipe is one-dimensional.

Source: Cengel et al. 2010

HEAT CONDUCTION IN SPHERES

Fourier's law of heat conduction for heat transfer through spherical layer can be expressed as:

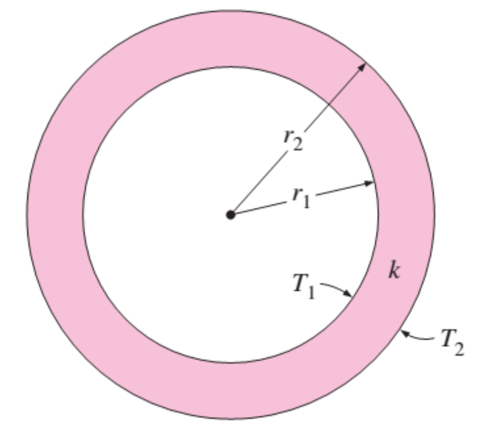
$$\dot{Q}_{cond,sph} = -kA\frac{dT}{dr}$$

$$\dot{Q}_{cond,sph} = \frac{T_1 - T_2}{R_{sph}} \qquad (W)$$

Thermal resistance of the cylindrical layer:

$$R_{sph} = \frac{r_2 - r_1}{4\pi r_1 r_2 k}$$

$$= \frac{\ln(\text{Outer radius}/\text{Inner radius})}{4\pi(Outer\ radius)(Inner\ radius)(Thermal\ conductivity)}$$



A long cylindrical pipe (or spherical shell) with specified inner and outer surface temperatures T_1 and T_2.

Source: Cengel et al. 2010

THERMAL RESISTANCE

Now consider steady one-dimensional heat flow through a cylindrical or spherical layer.

The rate of heat transfer under steady conditions can be expressed as

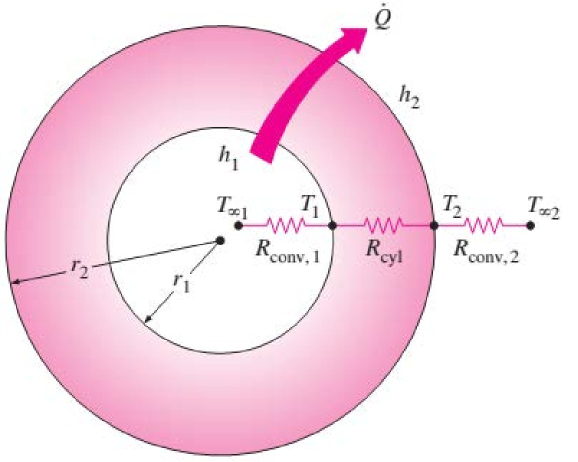
$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{total}}$$

For a cylindrical layer:

$$R_{total} = \frac{1}{(2\pi r_1 L)h_1} + \frac{\ln(r_2/r_1)}{2\pi L k} + \frac{1}{(2\pi r_2 L)h_2}$$

For a cylindrical layer:

$$R_{total} = \frac{1}{(4\pi r_1^2)h_1} + \frac{r_2 - r_1}{4\pi r_1 r_2 k} + \frac{1}{(4\pi r_2^2)h_2}$$



$R_{total} = R_{conv,1} + R_{cyl} + R_{conv,2}$

The thermal resistance network for a cylindrical (or spherical) shell subjected to convection from both the inner and the outer sides.

Source: Cengel et al. 2010

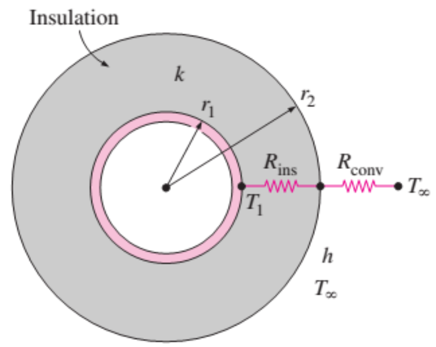
CRITICAL RADIUS OF INSULATION

The pipe is insulated with a material whose thermal conductivity is k and outer radius is r_2. Heat is lost from the pipe to the surrounding medium at temperature T_∞, with a convection heat transfer coefficient h. The rate of heat transfer:

$$\dot{\boldsymbol{Q}} = \frac{\boldsymbol{T_1 - T_\infty}}{R_{ins} + R_{conv}} = \frac{\boldsymbol{T_1 - T_\infty}}{\frac{\ln(r_2/r_1)}{2\pi Lk} + \frac{1}{h(2\pi r_2 L)}}$$

At which $\dot{\boldsymbol{Q}}$ reaches a maximum is determined from the requirement that $\mathrm{d}\dot{\boldsymbol{Q}}/\mathrm{d}r_2 = 0$ (zero slope). Now we can get **critical radius of insulation** by the differentiation and solving for r_2 $\longrightarrow$ $r_{cr,cylinder} = \frac{k}{h}$.

It can be repeated for a sphere, we get $r_{cr,sphere} = \frac{2k}{h}$

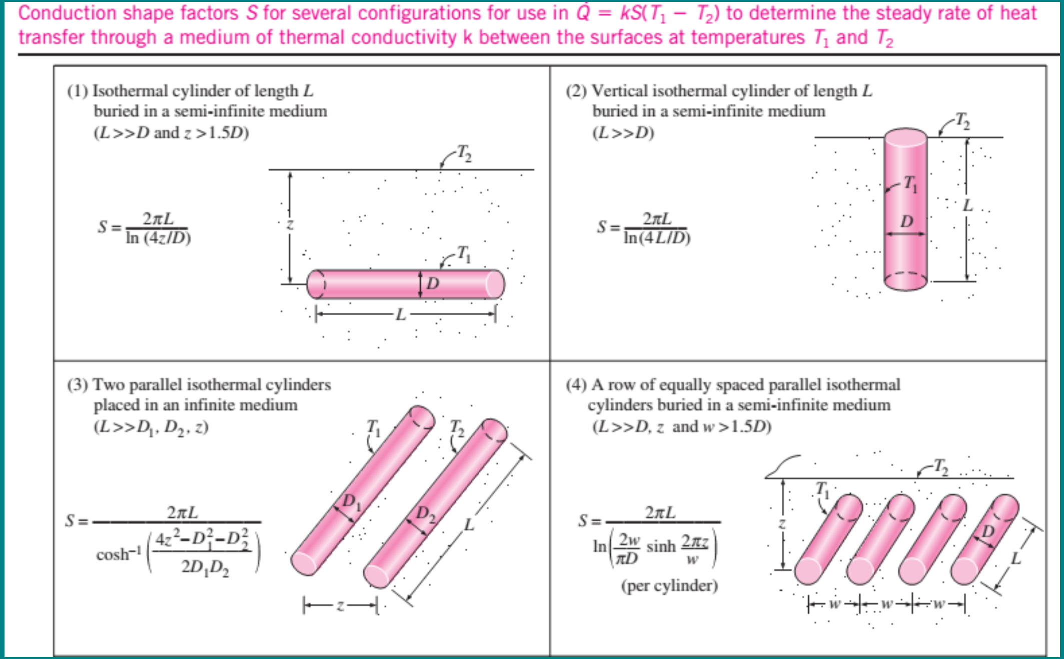


An insulated cylindrical pipe exposed to convection from the outer surface and the thermal resistance network associated with it.

HEAT TRANSFER IN COMMON CONFIGURATIONS

The steady rate of heat transfer between these two surfaces is expressed as:

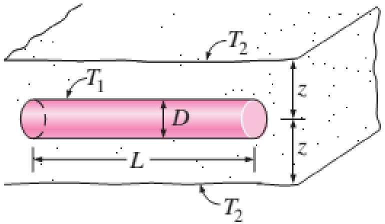
$Q = Sk(T_1 - T_2)$, where S is **conduction shape factor**



Conduction shape factor:

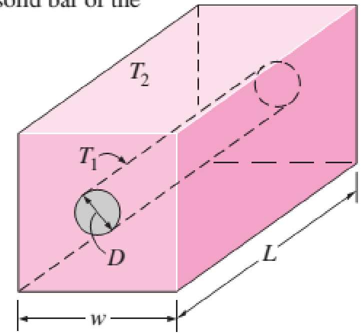
(5) Circular isothermal cylinder of length L in the midplane of an infinite wall ($z > 0.5D$)

$$S = \frac{2\pi L}{\ln(8z/\pi D)}$$



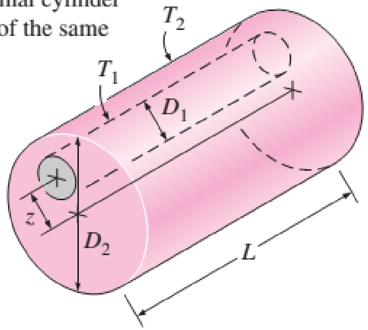
(6) Circular isothermal cylinder of length L at the center of a square solid bar of the same length

$$S = \frac{2\pi L}{\ln(1.08\, w/D)}$$



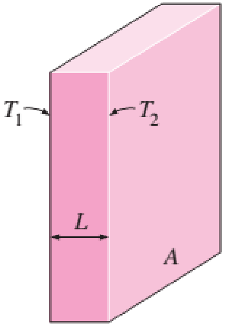
(7) Eccentric circular isothermal cylinder of length L in a cylinder of the same length ($L > D_2$)

$$S = \frac{2\pi L}{\cosh^{-1}\left(\dfrac{D_1^2 + D_2^2 - 4z^2}{2D_1D_2}\right)}$$



(8) Large plane wall

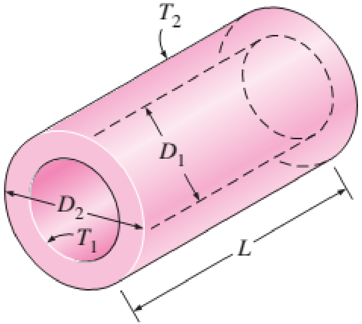
$$S = \frac{A}{L}$$



Conduction shape factor:

(9) A long cylindrical layer

$$S = \frac{2\pi L}{\ln (D_2/D_1)}$$



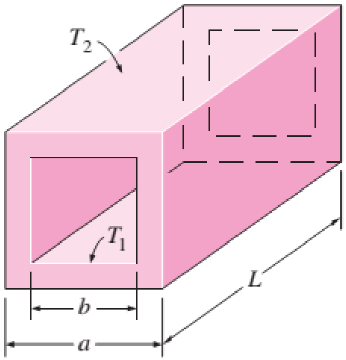
(10) A square flow passage

(a) For $a/b > 1.4$,

$$S = \frac{2\pi L}{0.93 \ln (0.948\, a/b)}$$

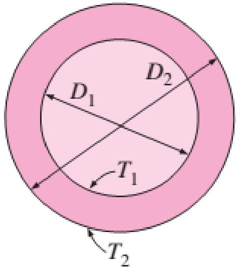
(b) For $a/b < 1.41$,

$$S = \frac{2\pi L}{0.785 \ln (a/b)}$$



(11) A spherical layer

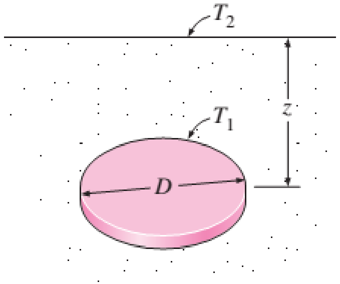
$$S = \frac{2\pi D_1 D_2}{D_2 - D_1}$$



(12) Disk buried parallel to the surface in a semi-infinite medium ($z >> D$)

$$S = 4D$$

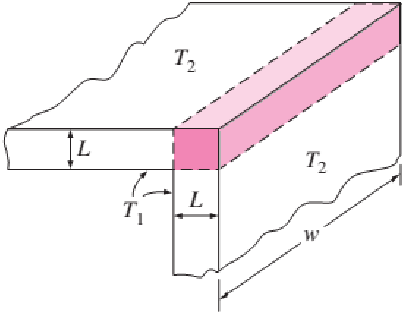
$(S = 2D$ when $z = 0)$



Conduction shape factor:

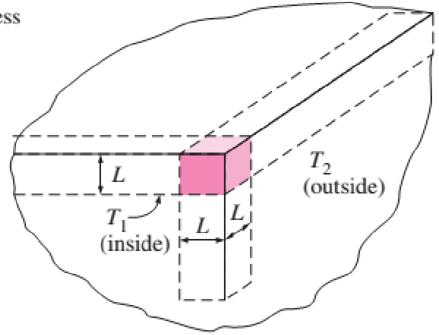
(13) The edge of two adjoining walls of equal thickness

$$S = 0.54\,w$$



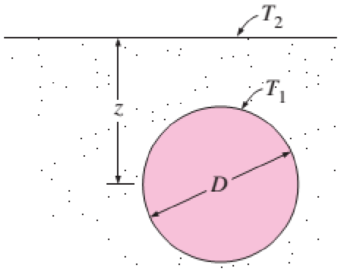
(14) Corner of three walls of equal thickness

$$S = 0.15L$$



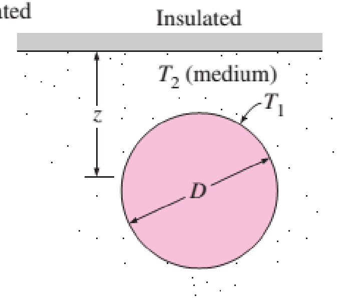
(15) Isothermal sphere buried in a semi-infinite medium

$$S = \frac{2\pi D}{1 - 0.25D/z}$$



(16) Isothermal sphere buried in a semi-infinite medium at T_2 whose surface is insulated

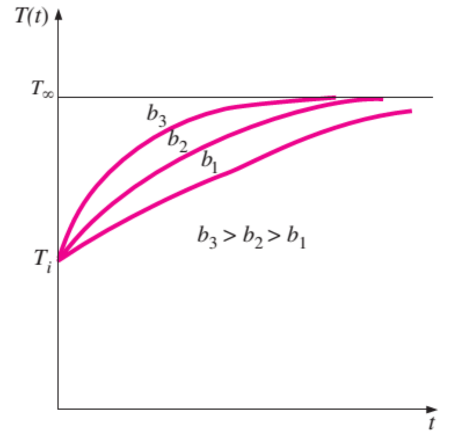
$$S = \frac{2\pi D}{1 + 0.25D/z}$$



TRANSIENT HEAT CONDUCTION

- Lumped System Analysis: interior temperature remains essentially uniform at all times during a heat transfer process

$$\begin{pmatrix} \text{Heat transfer into} \\ \text{the body during } dt \end{pmatrix} = \begin{pmatrix} \text{The increase in the energy} \\ \text{of the body during } dt \end{pmatrix}$$



The temperature of a lumped system approaches the environment temperature as time gets larger.

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt}$$

Where

$$b = \frac{hA_s}{\rho V C_p} \quad (1/s)$$

- The rate of convection heat transfer between the body and its environment at that time:

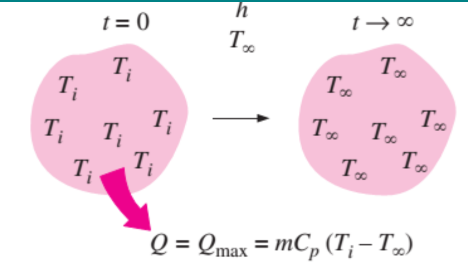
$$\dot{Q}(t) = hA_s[T(t) - T_\infty]$$

- The total amount of heat transfer between the body and the surrounding medium over the time interval $t = 0$ to t

$$Q = mC_p[T(t) - T_i]$$

- Therefore, the maximum heat transfer between the body and its surroundings

$$\boldsymbol{Q_{max}} = mC_p(T_\infty - T_i)$$



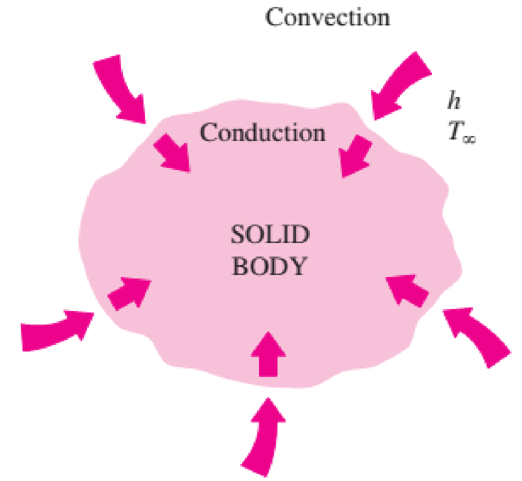
Heat transfer to or from a body reaches its maximum value when the body reaches the environment temperature.

Criteria for Lumped System Analysis

- Characteristic length: $L_c = \frac{V}{A_s}$
- **Biot number** : $Bi = \frac{hL_c}{k}$

$$\text{Bi} = \frac{h}{k/L_c}\frac{\Delta T}{\Delta T} = \frac{\text{Convection at the surface of the body}}{\text{Conduction within the body}}$$

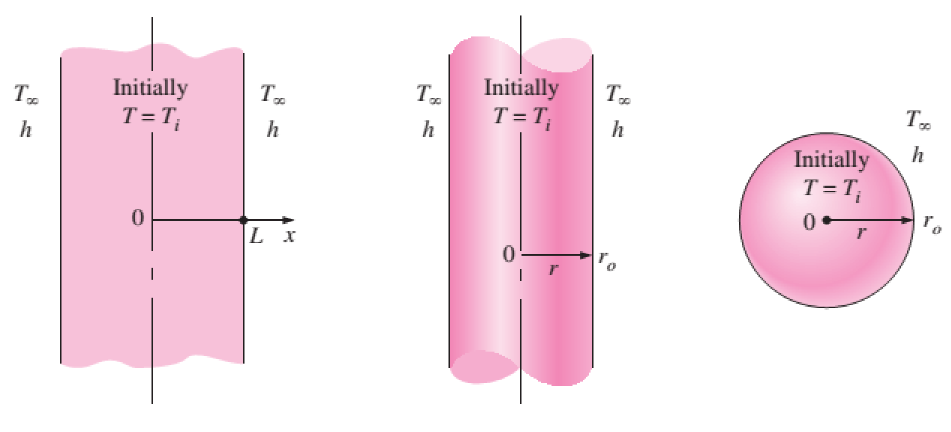
$$\text{Bi} = \frac{L_c/k}{1/h} = \frac{\text{Conduction resistance within the body}}{\text{Convection resistance at the surface of the body}}$$



$$\text{Bi} = \frac{\text{heat convection}}{\text{heat conduction}}$$

The Biot number can be viewed as the ratio of the convection at the surface to conduction within the body.

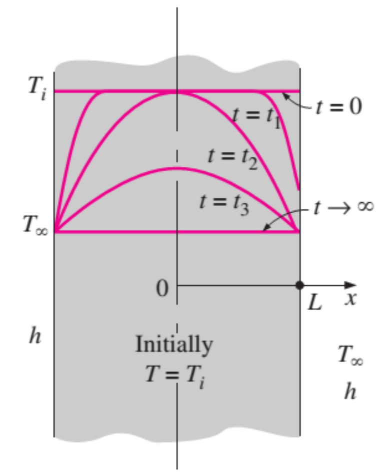
Transient Heat Conduction in Large Plane Walls, Long Cylinders, and Spheres with Spatial Effects



(*a*) A large plane wall (*b*) A long cylinder (*c*) A sphere

At time $t = 0$, each geometry is placed in a large medium that is at a constant temperature T_∞ and kept in that medium for $t > 0$. Heat transfer takes place between these bodies and their environments by convection with a uniform and constant heat transfer coefficient h.

The variation of the temperature profile with time in the plane wall



Transient temperature profiles in a plane wall exposed to convection from its surfaces for $T_i > T_\infty$.

The formulation of the problems for the determination of the one dimensional transient temperature distribution $T(x, t)$ in a wall results in a partial differential equation

The following dimensionless quantities:

Dimensionless temperature:	$\theta(x, t) = \dfrac{T(x, t) - T_\infty}{T_i - T_\infty}$	
Dimensionless distance from the center:	$X = \dfrac{x}{L}$	
Dimensionless heat transfer coefficient:	$\text{Bi} = \dfrac{hL}{k}$	**(Biot number)**
Dimensionless time:	$\tau = \dfrac{\alpha t}{L^2}$	**(Fourier number)**

The nondimensionalization enables us to present the temperature in terms of three parameters only: x, Bi, and τ

The solution of one-dimensional transient heat conduction using this **one term approximation**

Plane wall: $$\theta(x, t)_{\text{wall}} = \frac{T(x, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \cos(\lambda_1 x/L), \quad \tau > 0.2$$

Cylinder: $$\theta(r, t)_{\text{cyl}} = \frac{T(r, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} J_0(\lambda_1 r/r_o), \quad \tau > 0.2$$

Sphere: $$\theta(r, t)_{\text{sph}} = \frac{T(r, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \frac{\sin(\lambda_1 r/r_o)}{\lambda_1 r/r_o}, \quad \tau > 0.2$$

Noting that $\cos(0) = J_0(0) = 1$ and the limit of $(\sin x)/x$ is also 1, these relations simplify to the next ones at the center of a plane wall, cylinder, or sphere:

Center of plane wall ($x = 0$): $$\theta_{0,\ \text{wall}} = \frac{T_o - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$

Center of cylinder ($r = 0$): $$\theta_{0,\ \text{cyl}} = \frac{T_o - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$

Center of sphere ($r = 0$): $$\theta_{0,\ \text{sph}} = \frac{T_o - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$

Table of A_1 and λ_1 (functions of Bi)

Coefficients used in the one-term approximate solution of transient one-dimensional heat conduction in plane walls, cylinders, and spheres (Bi = hL/k for a plane wall of thickness $2L$, and Bi = hr_o/k for a cylinder or sphere of radius r_o)

	Plane Wall		Cylinder		Sphere	
Bi	λ_1	A_1	λ_1	A_1	λ_1	A_1
0.01	0.0998	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
0.1	0.3111	1.0161	0.4417	1.0246	0.5423	1.0298
0.2	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.3	0.5218	1.0450	0.7465	1.0712	0.9208	1.0880
0.4	0.5932	1.0580	0.8516	1.0931	1.0528	1.1164
0.5	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.6	0.7051	1.0814	1.0184	1.1345	1.2644	1.1713
0.7	0.7506	1.0918	1.0873	1.1539	1.3525	1.1978
0.8	0.7910	1.1016	1.1490	1.1724	1.4320	1.2236
0.9	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1.0	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2.0	1.0769	1.1785	1.5995	1.3384	2.0288	1.4793
3.0	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4.0	1.2646	1.2287	1.9081	1.4698	2.4556	1.7202
5.0	1.3138	1.2403	1.9898	1.5029	2.5704	1.7870
6.0	1.3496	1.2479	2.0490	1.5253	2.6537	1.8338
7.0	1.3766	1.2532	2.0937	1.5411	2.7165	1.8673
8.0	1.3978	1.2570	2.1286	1.5526	2.7654	1.8920
9.0	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10.0	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20.0	1.4961	1.2699	2.2880	1.5919	2.9857	1.9781
30.0	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40.0	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50.0	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100.0	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990
∞	1.5708	1.2732	2.4048	1.6021	3.1416	2.0000

Table of J_0 (the zeroth-order Bessel function of the first kind)

The zeroth- and first-order Bessel functions of the first kind

ξ	$J_0(\xi)$	$J_1(\xi)$
0.0	1.0000	0.0000
0.1	0.9975	0.0499
0.2	0.9900	0.0995
0.3	0.9776	0.1483
0.4	0.9604	0.1960
0.5	0.9385	0.2423
0.6	0.9120	0.2867
0.7	0.8812	0.3290
0.8	0.8463	0.3688
0.9	0.8075	0.4059
1.0	0.7652	0.4400
1.1	0.7196	0.4709
1.2	0.6711	0.4983
1.3	0.6201	0.5220
1.4	0.5669	0.5419
1.5	0.5118	0.5579
1.6	0.4554	0.5699
1.7	0.3980	0.5778
1.8	0.3400	0.5815
1.9	0.2818	0.5812
2.0	0.2239	0.5767
2.1	0.1666	0.5683
2.2	0.1104	0.5560
2.3	0.0555	0.5399
2.4	0.0025	0.5202
2.6	−0.0968	−0.4708
2.8	−0.1850	−0.4097
3.0	−0.2601	−0.3391
3.2	−0.3202	−0.2613

The change in the energy content of the body:

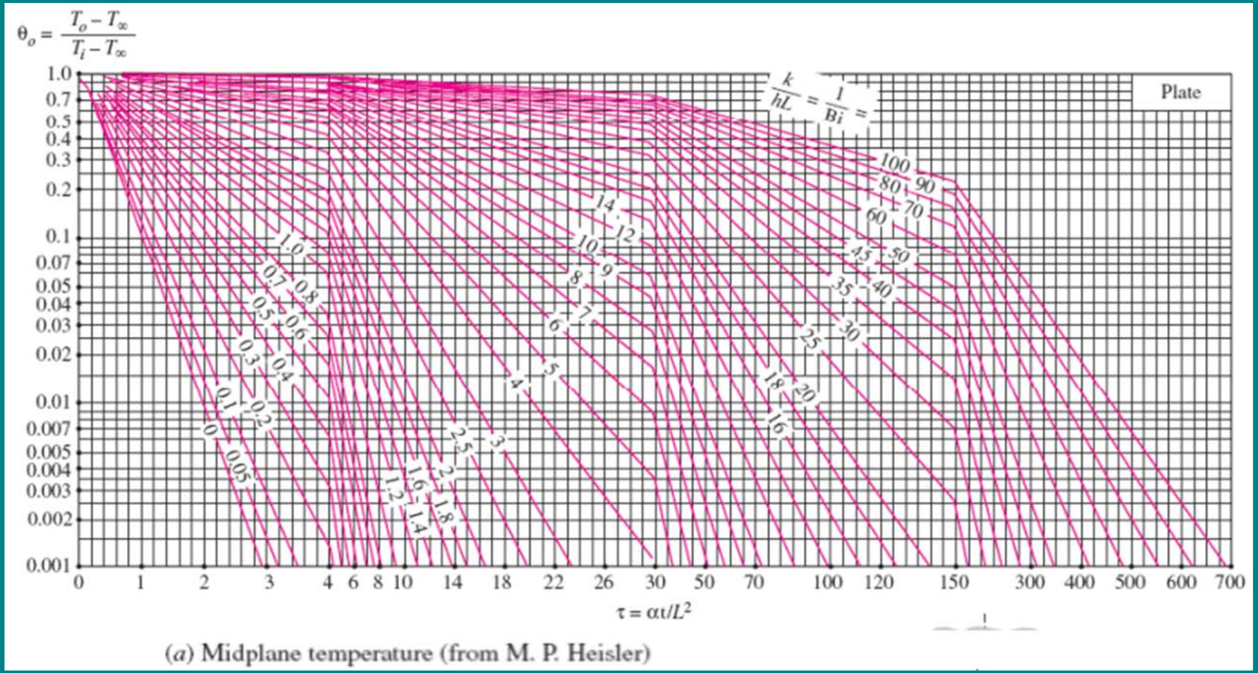
$$Q_{max} = mC_p(T_\infty - T_i) = \rho V C_p(T_\infty - T_i)$$

The fraction of heat transfer can abe determined from these relations, which are based on the one-term approximations:

Plane wall: $$\left(\frac{Q}{Q_{\max}}\right)_{\text{wall}} = 1 - \theta_{0,\,\text{wall}} \frac{\sin \lambda_1}{\lambda_1}$$

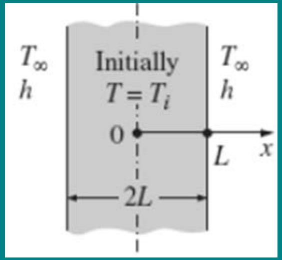
Cylinder: $$\left(\frac{Q}{Q_{\max}}\right)_{\text{cyl}} = 1 - 2\theta_{0,\,\text{cyl}} \frac{J_1(\lambda_1)}{\lambda_1}$$

Sphere: $$\left(\frac{Q}{Q_{\max}}\right)_{\text{sph}} = 1 - 3\theta_{0,\,\text{sph}} \frac{\sin \lambda_1 - \lambda_1 \cos \lambda_1}{\lambda_1^3}$$

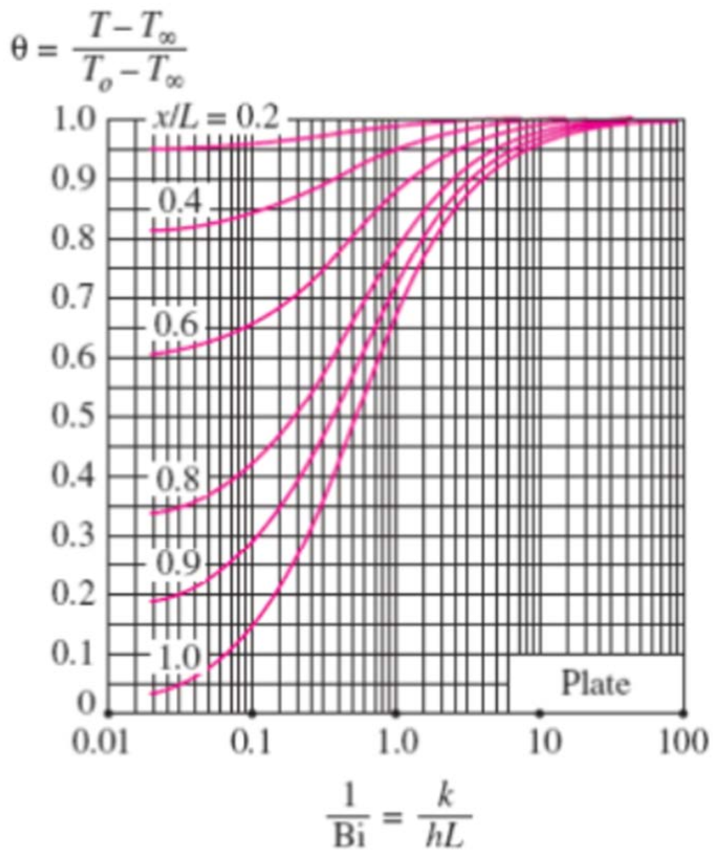


(a) Midplane temperature (from M. P. Heisler)

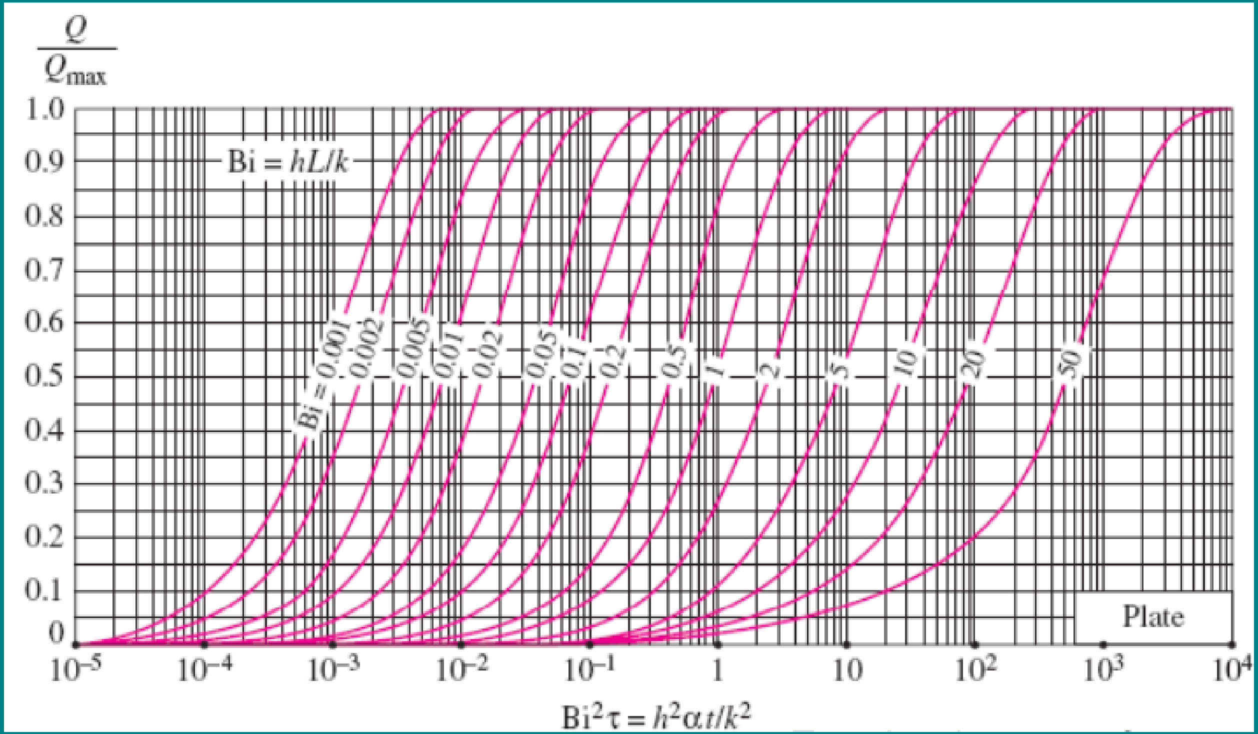
Transient mid-plane temperature chart for a plane wall of thickness 2L initiallt at uniform temperature T_i subjected to convection from both sides to an environment at temperature T_∞ with a convection coefficient of h



Transient temperature chart for a plane wall of thickness 2L initially at a uniform temperature T_i subjected to convection from both sides to an environment at temperature T_∞ with a convection coefficient of h.



(*b*) Temperature distribution (from M. P. Heisler)



Transient heat transfer chart for a plane wall of thickness 2L initially at a uniform temperature T_i subjected to convection from both sides to an environment at temperature T_∞ with a convection coefficient of h.

$$Q_{max} = m.C_p[T_\infty - T_i] = \rho V C_p[T_\infty - T_i] \text{ kJ}$$